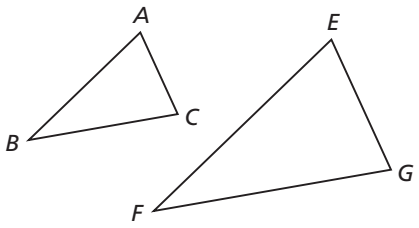
4.6 Similarity and Transformations

Essential Question When a figure is translated, reflected, rotated, or dilated in the plane, is the image always similar to the original figure?

Two figures are *similar figures* when they have the same shape but not necessarily the same size.



Similar Triangles

ATTENDING TO PRECISION

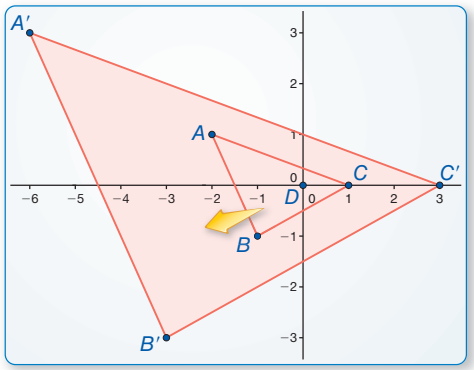
To be proficient in math, you need to use clear definitions in discussions with others and in your own reasoning.

EXPLORATION 1 Dilations and Similarity

Work with a partner.

a. Use dynamic geometry software to draw any triangle and label it $\triangle ABC$.

b. Dilate the triangle using a scale factor of 3. Is the image similar to the original triangle? Justify your answer.



Sample
Points
$A(-2, 1)$
$B(-1, -1)$
$C(1, 0)$
$D(0, 0)$
Segments
$AB = 2.24$
$BC = 2.24$
$AC = 3.16$
Angles
$m\angle A = 45°$
$m\angle B = 90°$
$m\angle C = 45°$

EXPLORATION 2 Rigid Motions and Similarity

Work with a partner.

a. Use dynamic geometry software to draw any triangle.

b. Copy the triangle and translate it 3 units left and 4 units up. Is the image similar to the original triangle? Justify your answer.

c. Reflect the triangle in the y-axis. Is the image similar to the original triangle? Justify your answer.

d. Rotate the original triangle 90° counterclockwise about the origin. Is the image similar to the original triangle? Justify your answer.

Communicate Your Answer

3. When a figure is translated, reflected, rotated, or dilated in the plane, is the image always similar to the original figure? Explain your reasoning.

4. A figure undergoes a composition of transformations, which includes translations, reflections, rotations, and dilations. Is the image similar to the original figure? Explain your reasoning.

4.6 Lesson

Core Vocabulary

similarity transformation, *p. 216*
similar figures, *p. 216*

What You Will Learn

- Perform similarity transformations.
- Describe similarity transformations.
- Prove that figures are similar.

Performing Similarity Transformations

A dilation is a transformation that preserves shape but not size. So, a dilation is a nonrigid motion. A **similarity transformation** is a dilation or a composition of rigid motions and dilations. Two geometric figures are **similar figures** if and only if there is a similarity transformation that maps one of the figures onto the other. Similar figures have the same shape but not necessarily the same size.

Congruence transformations preserve length and angle measure. When the scale factor of the dilation(s) is not equal to 1 or -1, similarity transformations preserve angle measure only.

EXAMPLE 1 Performing a Similarity Transformation

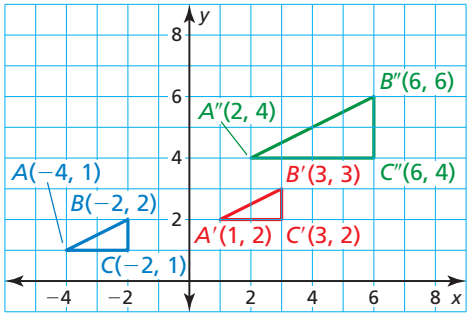
Graph $\triangle ABC$ with vertices $A(-4, 1)$, $B(-2, 2)$, and $C(-2, 1)$ and its image after the similarity transformation.

Translation: $(x, y) \rightarrow (x + 5, y + 1)$

Dilation: $(x, y) \rightarrow (2x, 2y)$

SOLUTION

Step 1 Graph $\triangle ABC$.



Step 2 Translate $\triangle ABC$ 5 units right and 1 unit up. $\triangle A'B'C'$ has vertices $A'(1, 2)$, $B'(3, 3)$, and $C'(3, 2)$.

Step 3 Dilate $\triangle A'B'C'$ using a scale factor of 2. $\triangle A''B''C''$ has vertices $A''(2, 4)$, $B''(6, 6)$, and $C''(6, 4)$.

Monitoring Progress

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1. Graph $\overline{CD}$ with endpoints $C(-2, 2)$ and $D(2, 2)$ and its image after the similarity transformation.

   Rotation: 90° about the origin

   Dilation: $(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$

2. Graph $\triangle FGH$ with vertices $F(1, 2)$, $G(4, 4)$, and $H(2, 0)$ and its image after the similarity transformation.

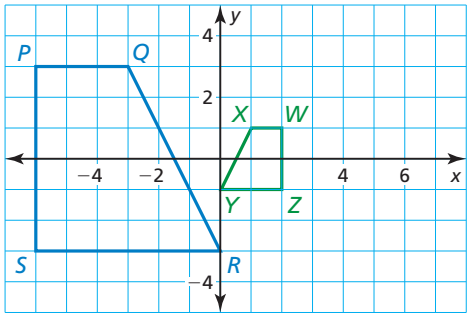
   Reflection: in the x-axis

   Dilation: $(x, y) \rightarrow (1.5x, 1.5y)$

Describing Similarity Transformations

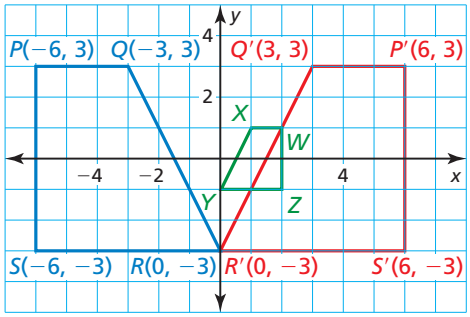
EXAMPLE 2 Describing a Similarity Transformation

Describe a similarity transformation that maps trapezoid $PQRS$ to trapezoid $WXYZ$.



SOLUTION

$\overline{QR}$ falls from left to right, and $\overline{XY}$ rises from left to right. If you reflect trapezoid $PQRS$ in the y-axis as shown, then the image, trapezoid $P'Q'R'S'$, will have the same orientation as trapezoid $WXYZ$.



Trapezoid $WXYZ$ appears to be about one-third as large as trapezoid $P'Q'R'S'$. Dilate trapezoid $P'Q'R'S'$ using a scale factor of $\frac{1}{3}$.

$$(x, y) \rightarrow \left(\tfrac{1}{3}x, \tfrac{1}{3}y\right)$$

$$P'(6, 3) \rightarrow P''(2, 1)$$

$$Q'(3, 3) \rightarrow Q''(1, 1)$$

$$R'(0, -3) \rightarrow R''(0, -1)$$

$$S'(6, -3) \rightarrow S''(2, -1)$$

The vertices of trapezoid $P''Q''R''S''$ match the vertices of trapezoid $WXYZ$.

▶ So, a similarity transformation that maps trapezoid $PQRS$ to trapezoid $WXYZ$ is a reflection in the y-axis followed by a dilation with a scale factor of $\frac{1}{3}$.

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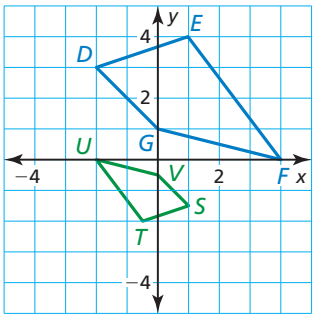
3. In Example 2, describe another similarity transformation that maps trapezoid $PQRS$ to trapezoid $WXYZ$.

4. Describe a similarity transformation that maps quadrilateral $DEFG$ to quadrilateral $STUV$.



Proving Figures Are Similar

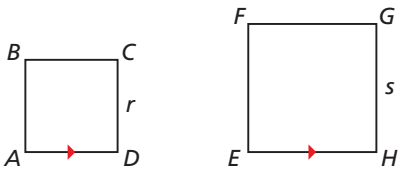
To prove that two figures are similar, you must prove that a similarity transformation maps one of the figures onto the other.

EXAMPLE 3 Proving That Two Squares Are Similar

Prove that square $ABCD$ is similar to square $EFGH$.

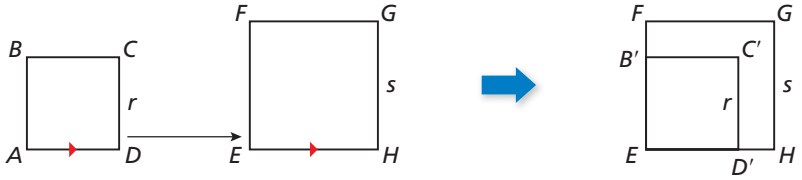
Given Square $ABCD$ with side length r, square $EFGH$ with side length s, $\overline{AD} \parallel \overline{EH}$

Prove Square $ABCD$ is similar to square $EFGH$.

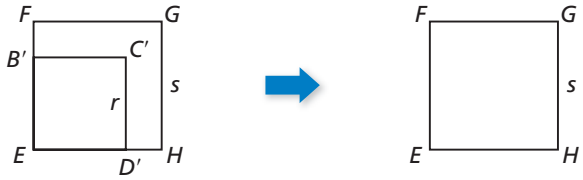


SOLUTION

Translate square $ABCD$ so that point A maps to point E. Because translations map segments to parallel segments and $\overline{AD} \parallel \overline{EH}$, the image of $\overline{AD}$ lies on $\overline{EH}$.

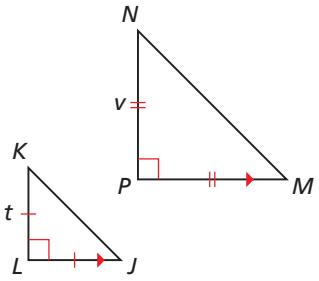


Because translations preserve length and angle measure, the image of $ABCD$, $EB'C'D'$, is a square with side length r. Because all the interior angles of a square are right angles, $\angle B'ED' \cong \angle FEH$. When $\overrightarrow{ED'}$ coincides with $\overrightarrow{EH}$, $\overrightarrow{EB'}$ coincides with $\overrightarrow{EF}$. So, $\overline{EB'}$ lies on $\overline{EF}$. Next, dilate square $EB'C'D'$ using center of dilation E. Choose the scale factor to be the ratio of the side lengths of $EFGH$ and $EB'C'D'$, which is $\frac{s}{r}$.



This dilation maps $\overline{ED'}$ to $\overline{EH}$ and $\overline{EB'}$ to $\overline{EF}$ because the images of $\overline{ED'}$ and $\overline{EB'}$ have side length $\frac{s}{r}(r) = s$ and the segments $\overline{ED'}$ and $\overline{EB'}$ lie on lines passing through the center of dilation. So, the dilation maps B' to F and D' to H. The image of C' lies $\frac{s}{r}(r) = s$ units to the right of the image of B' and $\frac{s}{r}(r) = s$ units above the image of D'. So, the image of C' is G.

▶ A similarity transformation maps square $ABCD$ to square $EFGH$. So, square $ABCD$ is similar to square $EFGH$.



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5. Prove that $\triangle JKL$ is similar to $\triangle MNP$.

Given Right isosceles $\triangle JKL$ with leg length t, right isosceles $\triangle MNP$ with leg length v, $\overline{LJ} \parallel \overline{PM}$

Prove $\triangle JKL$ is similar to $\triangle MNP$.

4.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What is the difference between *similar figures* and *congruent figures*?

2. **COMPLETE THE SENTENCE** A transformation that produces a similar figure, such as a dilation, is called a _____.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, graph $\triangle FGH$ with vertices $F(-2, 2)$, $G(-2, -4)$, and $H(-4, -4)$ and its image after the similarity transformation. *(See Example 1.)*

3. **Translation:** $(x, y) \rightarrow (x + 3, y + 1)$
   Dilation: $(x, y) \rightarrow (2x, 2y)$

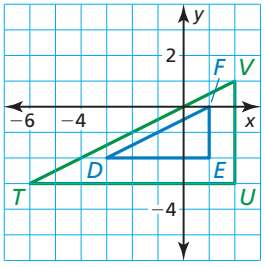
4. **Dilation:** $(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$
   Reflection: in the y-axis

5. **Rotation:** $90°$ about the origin
   Dilation: $(x, y) \rightarrow (3x, 3y)$

6. **Dilation:** $(x, y) \rightarrow \left(\frac{3}{4}x, \frac{3}{4}y\right)$
   Reflection: in the x-axis

In Exercises 7 and 8, describe a similarity transformation that maps the blue preimage to the green image. *(See Example 2.)*

7.



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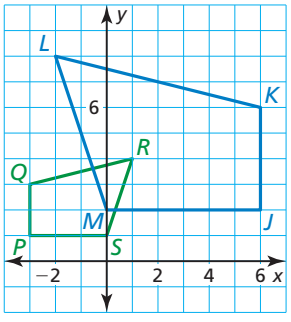


In Exercises 9–12, determine whether the polygons with the given vertices are similar. Use transformations to explain your reasoning.

9. $A(6, 0)$, $B(9, 6)$, $C(12, 6)$ and $D(0, 3)$, $E(1, 5)$, $F(2, 5)$

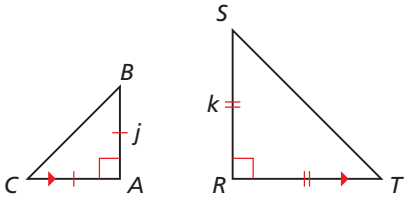
10. $Q(-1, 0)$, $R(-2, 2)$, $S(1, 3)$, $T(2, 1)$ and $W(0, 2)$, $X(4, 4)$, $Y(6, -2)$, $Z(2, -4)$

11. $G(-2, 3)$, $H(4, 3)$, $I(4, 0)$ and $J(1, 0)$, $K(6, -2)$, $L(1, -2)$

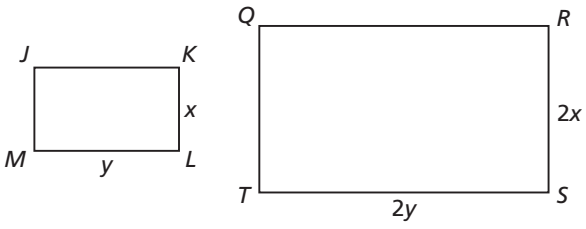
12. $D(-4, 3)$, $E(-2, 3)$, $F(-1, 1)$, $G(-4, 1)$ and $L(1, -1)$, $M(3, -1)$, $N(6, -3)$, $P(1, -3)$

In Exercises 13 and 14, prove that the figures are similar. *(See Example 3.)*

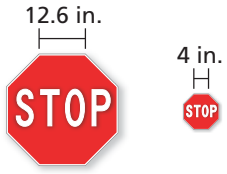
13. **Given** Right isosceles $\triangle ABC$ with leg length j, right isosceles $\triangle RST$ with leg length k, $\overline{CA} \parallel \overline{RT}$
    Prove $\triangle ABC$ is similar to $\triangle RST$.



14. **Given** Rectangle $JKLM$ with side lengths x and y, rectangle $QRST$ with side lengths $2x$ and $2y$
    Prove Rectangle $JKLM$ is similar to rectangle $QRST$.



15. MODELING WITH MATHEMATICS Determine whether the regular-sized stop sign and the stop sign sticker are similar. Use transformations to explain your reasoning.



16. ERROR ANALYSIS Describe and correct the error in comparing the figures.

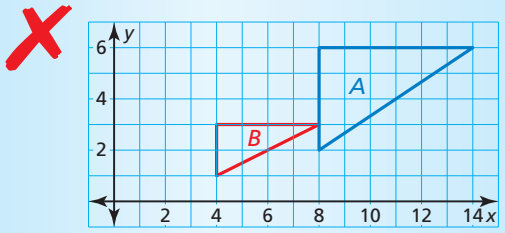
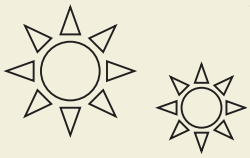


Figure A is similar to Figure B.

17. MAKING AN ARGUMENT A member of the homecoming decorating committee gives a printing company a banner that is 3 inches by 14 inches to enlarge. The committee member claims the banner she receives is distorted. Do you think the printing company distorted the image she gave it? Explain.

18. HOW DO YOU SEE IT? Determine whether each pair of figures is similar. Explain your reasoning.

a.



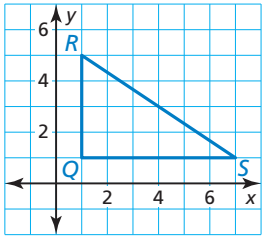
b.

19. ANALYZING RELATIONSHIPS Graph a polygon in a coordinate plane. Use a similarity transformation involving a dilation (where k is a whole number) and a translation to graph a second polygon. Then describe a similarity transformation that maps the second polygon onto the first.

20. THOUGHT PROVOKING Is the composition of a rotation and a dilation commutative? (In other words, do you obtain the same image regardless of the order in which you perform the transformations?) Justify your answer.

21. MATHEMATICAL CONNECTIONS Quadrilateral $JKLM$ is mapped to quadrilateral $J'K'L'M'$ using the dilation $(x, y) \rightarrow \left(\frac{3}{2}x, \frac{3}{2}y\right)$. Then quadrilateral $J'K'L'M'$ is mapped to quadrilateral $J''K''L''M''$ using the translation $(x, y) \rightarrow (x + 3, y - 4)$. The vertices of quadrilateral $J'K'L'M'$ are $J(-12, 0)$, $K(-12, 18)$, $L(-6, 18)$, and $M(-6, 0)$. Find the coordinates of the vertices of quadrilateral $JKLM$ and quadrilateral $J''K''L''M''$. Are quadrilateral $JKLM$ and quadrilateral $J''K''L''M''$ similar? Explain.

22. REPEATED REASONING Use the diagram.



a. Connect the midpoints of the sides of $\triangle QRS$ to make another triangle. Is this triangle similar to $\triangle QRS$? Use transformations to support your answer.

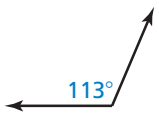
b. Repeat part (a) for two other triangles. What conjecture can you make?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Classify the angle as *acute*, *obtuse*, *right*, or *straight*. *(Section 1.5)*

23.



24.

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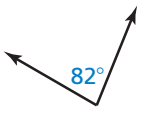

26.